

# Building $SO_{10}$ - models with $\mathbb{D}_4$ symmetry

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## Abstract

Using characters of finite group representations and monodromy of matter curves in F-GUT, we complete partial results in literature by building  $SO_{10}$  models with dihedral  $\mathbb{D}_4$  discrete symmetry. We first revisit the  $\mathbb{S}_4$ - and  $\mathbb{S}_3$ -models from the discrete group character view; then we extend the construction to  $\mathbb{D}_4$ . We find that there are three types of  $SO_{10} \times \mathbb{D}_4$  models depending on the ways the  $\mathbb{S}_4$ -triplets break down in terms of irreducible  $\mathbb{D}_4$ - representations:  $(\alpha)$  as  $\mathbf{1}_{+,-} \oplus \mathbf{1}_{+,-} \oplus \mathbf{1}_{-,+}$ ; or  $(\beta)$   $\mathbf{1}_{+,+} \oplus \mathbf{1}_{+,-} \oplus \mathbf{1}_{-,-}$ ; or also  $(\gamma)$   $\mathbf{1}_{+,-} \oplus \mathbf{2}_{0,0}$ . Superpotentials and other features are also given.

*Key words:*  $F$ - $SO_{10}$  models, discrete groups and characters,  $\mathbb{D}_4$  symmetry.

## 1 Introduction

In F-theory set-up, the study of GUT- models with discrete symmetries  $\Gamma$  given by  $\mathbb{S}_n$  permutation groups is generally done by using the splitting spectral method [1]-[7]; see also [8] and references therein. In the interesting case of  $SU_5 \times \Gamma$  models, the discrete  $\Gamma$ 's are mainly given by subgroups of the permutation symmetry  $\mathbb{S}_5$ ; in particular those subgroups like  $\mathbb{S}_4$ ,  $\mathbb{S}_3 \times \mathbb{S}_2$ ,  $\mathbb{S}_3, \mathbb{S}_2 \times \mathbb{S}_2$  and  $\mathbb{S}_2 \simeq \mathbb{Z}_2$ . To build  $SU_5$  GUT- models with exotic discrete groups like the alternating  $\mathbb{A}_4$  and the dihedral  $\mathbb{D}_4$ , one needs extra tools like Galois-theory [9, 10, 11, 12, 13, 14, 15]. In the case of  $SO_{10} \times \Upsilon$  models the situation is quite similar except that here the discrete  $\Upsilon$ 's are subgroups of the permutation symmetry  $\mathbb{S}_4$ .

In [9], an exhaustive study has been performed for several  $SO_{10} \times \Upsilon$  models, broken down to  $SU_5$ , by using splitting spectral method applied for the discrete subgroups  $\Upsilon$  given by

$\mathbb{S}_3$ ,  $\mathbb{Z}_3$ ,  $\mathbb{S}_2 \times \mathbb{S}_2$  and  $\mathbb{S}_2$ ; the common denominator of these  $\Upsilon$ 's is that they are subgroups of  $\mathbb{S}_4$ ; the Weyl group of the  $SU_4^\perp$  perpendicular symmetry to GUT gauge invariance in the  $E_8$  breaking down to  $SO_{10} \times SU_4^\perp$ . In this paper, we would like to complete the analysis of [9] and subsequent studies by considering the case of the order 8 dihedral symmetry  $\mathbb{D}_4$ . This discrete group is known to have two kinds of irreducible representations with dimensions 1 and 2; their multiplicities are read from the character relation  $8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$  teaching us the two following things: (i)  $\mathbb{D}_4$ - group has four 1-dim representations  $\mathbf{1}_{p,q}$ , including the trivial  $\mathbf{1}_{+,+}$ , the sign  $\mathbf{1}_{-,+}$  as wells as two others  $\mathbf{1}_{+,-}$  and  $\mathbf{1}_{-,-}$ ; and (ii) it has a unique 2-dim representation  $\mathbf{2}_{0,0}$  with vanishing character vector  $(0,0)$ . These irreducible representations are phenomenologically interesting; first because one of the four possible 1-dim representations of  $\mathbb{D}_4$  may be singled out to host the heaviest top- quark generation  $\mathbf{16}_3$ ; and the unique 2-dim representation of  $\mathbb{D}_4$  to accommodate the two other  $\mathbf{16}_{1,2}$  quark  $u$ - and  $c$ - families. This picture goes with the idea of [13] where Yukawa matrix  $\mathbf{Y}_{\{c,s,t\}}$  for the  $(u, c, t)$  quarks is approximated by a rank one matrix ( $rank \mathbf{Y} = 1$ ). There, the  $u, c$  quarks are taken in the massless approximation; and the third quark as a massive one. Moreover,  $\mathbb{D}_4$ - monodromy has also enough different singlets to accommodate the three quark generations independently; a property that may be used for studying superpotential prototypes leading to higher rank mass matrix with hierarchical eigenvalues.

The presentation is as follows: First we recall some useful aspects on  $SO_{10} \times \Upsilon$  models concerning standard  $\mathbb{S}_4$  and  $\mathbb{S}_3$  discrete symmetries. By using irreducible representations  $\mathbf{R}_i$  of these finite groups, we show how their character functions  $\chi_{\mathbf{R}_i}$  can be used to characterise the matter curve spectrum of these models. With these  $\chi_{\mathbf{R}_i}$  character tools at hand, we turn to build the above mentioned three  $SO_{10} \times \mathbb{D}_4$  models. We end this study by a conclusion and discussions on building superpotentials.

## 2 $SO_{10} \times \mathbb{S}_4$ model

We begin by recalling that in  $SO_{10} \times \mathbb{S}_4$  model of F-theory GUTs, matter curves carry quantum numbers in the  $SO_{10} \times SU_4^\perp$  representations following from the breaking of the  $E_8$  gauge symmetry of F-theory on elliptically fibered Calabi-Yau fourfold (CY4) with 7-brane wrapping  $\mathcal{S}_{GUT}$ ; the so called GUT surface [1, 9, 12, 13, 16],

$$\begin{array}{ccc} E & \rightarrow & \text{CY4} \\ & \downarrow & \\ & \mathcal{B}_3 & \end{array}, \quad \mathcal{B}_3 \supset \mathcal{S}_{GUT}$$

In this construction, the base  $\mathcal{B}_3$  is a complex 3 dim manifold containing  $\mathcal{S}_{GUT}$ ; and the fiber E is given by a particular Tate representation of the complex elliptic curve, namely

$$y^2 = x^3 + b_5xyz + b_4x^2z + b_3yz^2 + b_2xz^3 + b_0z^5 \quad (2.1)$$

where the homology classes  $[x], [y], [z], [b_k]$ , associated with the holomorphic sections  $x, y, z$  and  $b_k$ , are expressed in terms of the Chern class  $c_1 = c_1(\mathcal{S}_{GUT})$  of the tangent bundle of the  $\mathcal{S}_{GUT}$  surface; and the Chern class  $-t$  of the normal bundle  $\mathcal{N}_{\mathcal{S}_{GUT}|\mathcal{B}_3}$  as follows

$$\begin{aligned} [y] &= 3(c_1 - t) & , & & [z] &= -t \\ [x] &= 2(c_1 - t) & , & & [b_k] &= (6c_1 - t) - kc_1 \end{aligned} \quad (2.2)$$

Recall also that in  $SO_{10}$  models building with discrete symmetries  $\Gamma$ , the  $E_8$  symmetry of underlying F-theory on CY4 is broken down to  $SO_{10} \times \Gamma$ ; where  $SO_{10}$  is the GUT gauge symmetry and  $\Gamma$  a discrete monodromy group contained in  $\mathbb{S}_4$ . This symmetric  $\mathbb{S}_4$  is isomorphic to the Weyl symmetry group of the perpendicular  $SU_4^\perp$  to GUT symmetry inside  $E_8$ ; that is the commutant of  $SO_{10}$  in the exceptional  $E_8$  group of F-theory GUTs. From the decomposition of the  $E_8$  adjoint representations down to  $SO_{10} \times SU_4^\perp$  ones namely

$$\begin{aligned} \mathbf{248} \quad \rightarrow \quad & (\mathbf{45}, \mathbf{1}_\perp) \oplus (\mathbf{1}, \mathbf{15}_\perp) \oplus \\ & (\mathbf{16}, \mathbf{4}_\perp) \oplus (\overline{\mathbf{16}}, \overline{\mathbf{4}}_\perp) \oplus (\mathbf{10}, \mathbf{6}_\perp) \end{aligned} \quad (2.3)$$

we learn the matter content of  $SO_{10} \times SU_4^\perp$  theory; and then of the desired curves spectrum of the  $SO_{10} \times \mathbb{S}_4$  model. This spectrum is given by the following  $SO_{10}$  multiplets, labeled by four weights  $t_i$  of the fundamental representation of  $SU_4^\perp$ ,

$$\mathbf{16}_{t_i}, \quad \mathbf{16}_{-t_i}, \quad \mathbf{10}_{t_i+t_j}, \quad \mathbf{1}_{t_i-t_j} \quad (2.4)$$

with the traceless condition

$$t_1 + t_2 + t_3 + t_4 = 0 \quad (2.5)$$

The discrete symmetry  $\mathbb{S}_4$  acts by permutation of the  $t_i$  curves; it leaves stable the constraint (2.5) as well as observable of the model.

In  $SO_{10} \times \mathbb{S}_4$  theory, the 16-plets and the 10-plets are thought of as *reducible* multiplets of the  $\mathbb{S}_4$  Weyl symmetry of  $SU_4^\perp$ ; however from the view of GUT phenomenology, this  $\mathbb{S}_4$  monodromy symmetry must be broken since it treats the three GUT- generations on equal footing. But, for later use, we propose to study first the structure of the  $\mathbb{S}_4$  based model and some of its basic properties; then turn back to study the breaking of  $\mathbb{S}_4$  down to the subgroup  $\mathbb{S}_3$ ; and after to the  $\mathbb{D}_4$  we are interested in this paper.

## 2.1 Spectrum in $\mathbb{S}_4$ model

The  $16_{t_i}$  components of the four 16-plets and the  $10_{t_i+t_j}$  of the six 10-plets are related among themselves by  $\mathbb{S}_4$  monodromy; they are respectively given by the zeros of the holomorphic sections  $b_4$  and  $d_6$ , describing the intersections of the spectral covers  $\mathcal{C}_4 = 0$  and  $\mathcal{C}_6 = 0$  with the GUT surface  $s = 0$ . The defining eqs of these spectral covers are as shown on the following table,

matters curves	weight	$\mathbb{S}_4$	homology	holomorphic section
$16_{t_i}$	$t_i$	4	$\eta - 4c_1$	$b_4 = b_0 \prod_{i=1}^4 t_i = 0$
$10_{t_i+t_j}$	$t_i + t_j$	6	$\eta' - 6c_1$	$d_6 = d_0 \prod_{j>i=1}^4 T_{ij} = 0$

(2.6)

with [9, 12, 17]

$$\begin{aligned} \mathcal{C}_4 &= \sum_{k=0}^4 b_k s^{5-k} = b_0 \prod_{i=1}^4 (s - t_i) \\ \mathcal{C}_6 &= \sum_{k=0}^6 d_k s^{6-k} = d_0 \prod_{j>i=1}^6 (s - T_{ij}) \end{aligned} \quad (2.7)$$

and  $T_{ij} \equiv t_i + t_j$ . The homology class  $[b_4] = \eta - 4c_1$  is obtained by using the relation  $[b_0] + \sum [t_i]$  together with the canonical class  $[b_0] = \eta$  and  $[t_i] = -c_1$ ; the same feature leads to  $[d_6] = \eta' - 6c_1$ .

The  $\mathbb{S}_4$  invariance of  $\mathcal{C}_4$  is manifestly exhibited

$$\mathcal{C}_4 = \frac{b_0}{4!} \sum_{\sigma \in \mathbb{S}_4} \prod_{i=1}^4 (s - t_{\sigma(i)}) \quad (2.8)$$

A similar relation is also valid for  $\mathcal{C}_6$ . In what follows, and to fix ideas, we will think of the defining eqs of the spectral covers  $\mathcal{C}_4$  and  $\mathcal{C}_6$  as given by the last column of eqs(2.7) involving  $b_0$  and the four roots  $t_i$  (resp  $d_0$  and the six  $T_{ij}$ 's).

## 2.2 Characters in $SO_{10} \times \mathbb{S}_4$ model

The discrete symmetry group of the  $SO_{10} \times \mathbb{S}_4$  model has  $24$  elements that can be arranged into five conjugacy classes  $\mathfrak{C}_1, \dots, \mathfrak{C}_5$  with representatives given by p-cycles  $(12\dots p)$  and products type  $(\alpha\beta)(\gamma\delta)$  as shown on table (2.9). This finite group has five irreducible representations  $\mathbf{R}_1, \dots, \mathbf{R}_5$  whose dimensions can be learnt from the usual relation  $24 = 1^2 + 1^{2'} + 2^2 + 3^2 + 3^{2'}$  linking group order to the square of  $\dim \mathbf{R}_i$ ; their characters

$\chi_{\mathbf{R}_j}$ , describing mappings:  $\mathbb{S}_4 \rightarrow \mathbb{C}$ , are conjugacy class functions; associating to each class  $\mathfrak{C}_i$  the numbers  $\chi_{ij} = \chi_{\mathbf{R}_j}(\mathfrak{C}_i)$ ; whose explicit expressions are as given below

$\mathfrak{C}_i \backslash \text{irrep } \mathbf{R}_j$	$\chi_I$	$\chi_{\mathbf{3}'}$	$\chi_{\mathbf{2}}$	$\chi_{\mathbf{3}}$	$\chi_{\epsilon}$	order
$\mathfrak{C}_1 \equiv e$	1	3	2	3	1	1
$\mathfrak{C}_2 \equiv (\alpha\beta)$	1	-1	0	1	-1	6
$\mathfrak{C}_3 \equiv (\alpha\beta)(\gamma\delta)$	1	-1	2	-1	1	3
$\mathfrak{C}_4 \equiv (\alpha\beta\gamma)$	1	0	-1	0	1	8
$\mathfrak{C}_5 \equiv (\alpha\beta\gamma\delta)$	1	1	0	-1	-1	6

(2.9)

There are various manners to approach the properties of reducible  $\mathcal{R}$  and irreducible  $\mathbf{R}_i$  representations of the permutation group  $\mathbb{S}_4$ ; the natural way may be the one using graphic methods based on the Young diagrams [18]; where the five irreducible representations are represented by 5 diagrams as follows

$$1 : \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} , \quad 2 : \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} , \quad 3 : \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \quad (2.10)$$

and

$$3' : \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} , \quad 1' : \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \quad (2.11)$$

But here we will deal with  $\mathbb{S}_4$  by focussing on the properties of its three generators  $(a, b, c)$ ; these are basic operators of the  $\mathbb{S}_4$  group; and are generally chosen as given by the transpositions  $\tau_1 = (12)$ ,  $\tau_2 = (23)$  and  $\tau_3 = (34)$ ; they can be also taken as follows

$$\begin{aligned} A &= (12) = \tau_1 \\ B &= (123) = \tau_1\tau_2 \\ C &= (1234) = \tau_1\tau_2\tau_3 \end{aligned} \quad (2.12)$$

These three generators are not independent seen that they are non commuting operators,  $AB \neq BA$ ,  $AC \neq CA$ ,  $BC \neq CB$ ; a feature that makes extracting total information a complicated matter; so we will restrict to use their representation characters  $\chi_{\mathbf{R}_i}(A)$ ,  $\chi_{\mathbf{R}_i}(B)$  and  $\chi_{\mathbf{R}_i}(C)$  with  $\mathbf{R}_i$  given by (2.10-2.11); we also use the sums of  $\chi_{\mathbf{R}_i}(G)$  and their products. To that purpose, let us briefly recall some useful tools on  $\mathbb{S}_4$ -representations that we illustrate on  $SO_{10} \times \mathbb{S}_4$ -theory. First notice that the 4-dim permutation module  $\mathcal{V}_4$  of the group  $\mathbb{S}_4$  is interpreted in  $SO_{10} \times \mathbb{S}_4$ -model in terms of the four matter curves  $\mathbf{16}_i$ ; its 6-dim antisymmetric tensor product  $\mathcal{V}_6 = (\mathcal{V}_4 \otimes \mathcal{V}_4)_{\text{antisym}}$  as describing

the Higgs curves  $\mathbf{16}_{[ij]}$ ; and the  $\mathcal{V}_4 \otimes \mathcal{V}_4^*$  tensor product module is associated with flavons  $\vartheta_{ij}$ . All these spaces are reducible under  $\mathbb{S}_4$ ; and so permutation operators of  $\mathbb{S}_4$  can be generically decomposed as sums

$$n_1 \mathbf{1} \oplus n'_1 \mathbf{1}' \oplus n_2 \mathbf{2} \oplus n'_3 \mathbf{3} \oplus n'_3 \mathbf{3} \quad (2.13)$$

on the irreducible modules with some  $n_i$  multiplicities; for example

$$\begin{aligned} \mathbf{4} &= \mathbf{1} \oplus \mathbf{3} \\ \mathbf{6} &= \mathbf{3} \oplus \mathbf{3}' \end{aligned} \quad \leftrightarrow \quad \begin{aligned} \mathcal{V}_4 &= \mathcal{V}_1 \oplus \mathcal{V}_3 \\ \mathcal{V}_6 &= \mathcal{V}'_3 \oplus \mathcal{V}''_3 \end{aligned} \quad (2.14)$$

with  $\mathbf{6} = (\mathbf{4} \otimes \mathbf{4})_{\text{antisym}}$ . In practice, the use of irreducible representations as in (2.13) for  $SO_{10} \times \mathbb{S}_4$  modeling is achieved by starting from eqs(2.6); then look for an adequate basis vector change of the weight  $\{t_1, t_2, t_3, t_4\}$  into a new  $\{x_0, x_1, x_2, x_3\}$  basis where one of the components, say  $x_0$ , has the form

$$x_0 = \frac{1}{4} (t_1 + t_2 + t_3 + t_4) \quad (2.15)$$

this sum of weights is associated with the trivial representation  $\mathbf{1}$ , the completely symmetric representation; it is invariant under  $\mathbb{S}_4$ ; but also under all its subgroups including the  $\mathbb{S}_3$  and the  $\mathbb{D}_4$  we will encounter below. The three other  $(x_1, x_2, x_3) = \vec{x}$  transform as an irreducible triplet  $\mathbf{3}$  under of  $\mathbb{S}_4$ ; but differently under subgroups  $\mathbb{S}_3$  and  $\mathbb{D}_4$ ; their explicit expressions  $\vec{x} = \vec{x}(t_1, t_2, t_3, t_4)$  are given by;

$$\begin{aligned} x_1 &= \frac{1}{4} (t_1 + t_2 - t_3 - t_4) \\ x_2 &= \frac{1}{4} (t_1 - t_2 + t_3 - t_4) \\ x_3 &= \frac{1}{4} (t_1 - t_2 - t_3 + t_4) \end{aligned} \quad (2.16)$$

with (2.5) mapped to

$$x_0 + x_1 + x_2 + x_3 = t_1 \quad (2.17)$$

With the basis change of  $\{|t_\mu\rangle\}$  into  $\{|x_0\rangle; |x_i\rangle\}$ , the four matter  $16_{t_\mu}$  and the six Higgs  $10_{t_\mu+t_\nu}$  multiplets get splitted like

$$16_{t_\mu} \rightarrow \begin{pmatrix} 16'_0 \\ 16'_i \end{pmatrix}_{|x\rangle}, \quad 10_{t_\mu+t_\nu} \rightarrow \begin{pmatrix} 10'_{[ij]} \\ 10'_i \end{pmatrix}_{|x \otimes x\rangle} \quad (2.18)$$

In matrix notation with basis  $\{|x_0\rangle; |x_i\rangle\}$ , permutation operators  $\mathbf{P}_\sigma^{(4)}$  acting on  $\mathcal{V}_4$  and operators  $\mathbf{P}_\sigma^{(6)}$  on  $\mathcal{V}_6$  have the representation

$$\mathbf{P}_\sigma^{(4)} = \begin{pmatrix} 1 & 0_{1 \times 3} \\ 0_{3 \times 1} & \mathcal{P}_\sigma^{(3)} \end{pmatrix}, \quad \mathbf{P}_\sigma^{(6)} = \begin{pmatrix} \mathcal{P}_\sigma^{(3)} & 0_{3 \times 3} \\ 0_{3 \times 3} & \mathcal{P}_\sigma^{(3')} \end{pmatrix} \quad (2.19)$$

To describe the new matter curves  $\{16'_0, 16'_i\}$  and  $\{10'_i, 10'_{[ij]}\}$  we shall use the character  $\chi_{\mathbf{R}_i}$  of the irreducible representations of  $\mathbb{S}_4$  given by (2.9); in particular the character of the  $(A, B, C)$  generators of  $\mathbb{S}_4$ ; and which we denote as  $\chi_{\mathbf{R}_i}^{(G)}$  where  $G$  stands for  $(A, B, C)$ ,

$\chi_{\mathbf{R}_i}^{(G)}$	$\chi_I$	$\chi_{\mathbf{3}'}$	$\chi_{\mathbf{2}}$	$\chi_{\mathbf{3}}$	$\chi_{\epsilon}$
$A$	1	-1	0	1	-1
$B$	1	0	-1	0	1
$C$	1	1	0	-1	-1

(2.20)

Because of a standard feature of the trace of direct sum of matrices namely  $Tr(\mathcal{A} \oplus \mathcal{B}) = Tr\mathcal{A} + Tr\mathcal{B}$ , we also use the following property relating the characters of reducible  $\mathcal{R}$  representations to their  $\mathbf{R}_i$  irreducible components

$$\mathcal{R} = n_1 \mathbf{R}_1 \oplus n_2 \mathbf{R}_2 \quad \Rightarrow \quad \chi_{\mathcal{R}}^{(G)} = n_1 \chi_{\mathbf{R}_1}^{(G)} + n_2 \chi_{\mathbf{R}_2}^{(G)} \quad (2.21)$$

For the example of the quartet  $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$  of the group  $\mathbb{S}_4$ , we have the relation  $\chi_{\mathbf{1} \oplus \mathbf{3}}^{(G)} = \chi_{\mathbf{1}}^{(G)} + \chi_{\mathbf{3}}^{(G)}$ ; and remembering the interpretation of the characters in terms of fix points of the permutation symmetry; the character vector

$$\chi_{\mathbf{4}}^{(G)} = (2, 1, 0) \equiv \chi_{\mathbf{1} \oplus \mathbf{3}}^{(G)} \quad (2.22)$$

splits therefore as the sum of two terms:  $\chi_{\mathbf{1}}^{(G)} = (1, 1, 1)$  and  $\chi_{\mathbf{3}}^{(G)} = (1, 0, -1)$ ; in agreement with the character table (2.20). Applying also this property to  $\mathbf{3} \oplus \mathbf{3}' = \mathbf{6}$ , we have  $\chi_{\mathbf{6}}^{(G)} = \chi_{\mathbf{3}}^{(G)} + \chi_{\mathbf{3}'}^{(G)}$ ; from which we learn the value of the character of the reducible 6-dimensional representation of  $\mathbb{S}_4$ ; it vanishes identically. Therefore, the matter curves spectrum of the  $SO_{10} \times \mathbb{S}_4$  model in the  $\{|x\rangle\}$  and  $\{|x \otimes x'\rangle\}$  bases reads as follows

matters curves	$\mathbb{S}_4$	homology	$U(1)_x$ flux	character $\chi_{\mathbf{R}}^{(G)}$
$16'_0$	<b>1</b>	$-c_1$	0	$(1, 1, 1)$
$16'_i$	<b>3</b>	$\eta - 3c_1$	0	$(1, 0, -1)$
$10'_{[ij]}$	<b>3'</b>	$-3c_1$	0	$(-1, 0, 1)$
$10'_i$	<b>3</b>	$\eta' - 3c_1$	0	$(1, 0, -1)$

(2.23)

where the homology classes of the new matter curves  $16'_0, 16'_i$ ; and  $10'_{[ij]}, 10'_i$  are derived from the homology of the reducible curve multiplets  $16_{t_\mu}, 10_{t_\mu+t_\nu}$  of table eqs(2.6) as follows

$$\begin{aligned} [16_{t_\mu}] &= [16'_0] + [16'_i] \\ [10_{t_\mu+t_\nu}] &= [10'_{[ij]}] + [10'_i] \end{aligned} \quad (2.24)$$

Notice that though the matter curves spectrum looks splitted, it is still invariant under  $\mathbb{S}_4$  monodromy. It is just a property of the  $|x_{0,i}\rangle$  frame where the completely symmetric  $x_0$

component weight, the centre of weights, is thought of as the origin of the frame. Notice also that the 15 flavons of the  $SO_{10} \times \mathbb{S}_4$  model split as  $3 \oplus 12$ ; with the 12 charged ones splitting like  $12 = 6 \oplus 6^*$  where the 6, currently denoted as  $\vartheta_{+(t_\mu-t_\nu)}$  with  $\mu < \nu$ ; and the other  $6^*$  with  $\vartheta_{-(t_\mu-t_\nu)}$ . Because of the real values of the characters (2.9), the complex adjoints  $6^* = 3^* \oplus 3'^*$  may be thought in terms of dual representations namely  $3^* \sim 3'$  and  $3'^* \sim 3$ ; so the characters for the 12 charged flavons read as

flavons	character $\chi_{\mathbf{R}}^{(G)}$
$1_i$	$(1, 0, -1)$
$1_{[ij]}$	$(-1, 0, 1)$
$1_{[ij]}$	$(-1, 0, 1)$
$1_i$	$(1, 0, -1)$

(2.25)

### 3 Building $SO_{10} \times \mathbb{D}_4$ models

First we describe the key idea of our method that we illustrate on the example of  $SO_{10} \times \mathbb{S}_3$  model; seen that  $\mathbb{S}_3$  is a subgroup of  $\mathbb{S}_4$  just as  $\mathbb{D}_4$ . Then we turn to study the  $SO_{10} \times \mathbb{D}_4$  theory; and derive its matter curves spectrum and their characters; comments on the superpotentials  $W$  of  $\mathbb{D}_4$  models and others aspects will be given in conclusion and discussion section.

#### 3.1 Revisiting $SO_{10} \times \mathbb{S}_3$ model

To engineer  $SO_{10} \times \Gamma$  models with discrete symmetries  $\Gamma$  contained in  $\mathbb{S}_4$ , we have to break the  $\mathbb{S}_4$  symmetry down to its subgroup  $\Gamma$ . Seen that  $\mathbb{S}_4$  has 30 subgroups; one ends with a proleariat of  $SO_{10}$  models with discrete symmetries; some of these monodromies are related amongst others by similarity transformations. For example,  $\mathbb{S}_4$  has four  $\mathbb{S}_3$  subgroups obtained by fixing one of the four  $t_i$  roots of the spectral cover  $\mathcal{C}_4$  of eq(2.7); but these  $\mathbb{S}_3$  groups are isomorphic to each other; and so it is enough to consider just one of them; say the one fixing the weight  $t_4$ ; and permuting the other three  $t_1, t_2, t_3$ ; i.e:

$$\begin{aligned} \sigma(t_4) &= t_4 \\ \sigma(\{t_1, t_2, t_3\}) &= \{t_1, t_2, t_3\} \end{aligned} \tag{3.1}$$

leading to  $\sigma \in \mathbb{S}_4/\mathcal{J}$ , with  $\mathcal{J} = \langle t_4 - \sigma(t_4) \rangle$ ; it is isomorphic to  $\mathbb{S}_3$ . Let us describe rapidly our method of engineering  $SO_{10} \times \mathbb{S}_3$  model; and extend later this construction to the case of the order 8 dihedral  $\mathbb{D}_4$ .

Starting from the matter spectrum of the  $SO_{10} \times \mathbb{S}_4$  model (2.6), we can derive the



properties of the matter curves of the  $SO_{10} \times \mathbb{S}_3$  model by using the breaking pattern

$$\mathbb{S}_4 \rightarrow \mathbb{S}_3 \times \mathbb{S}_1 \quad (3.2)$$

where  $\mathbb{S}_1$  factor is associated with the fixed weight  $t_4$ ; it may be interpreted in terms of a conserved  $U_1^\perp$  symmetry inside  $SU_4^\perp$ . The descent from  $\mathbb{S}_4$  to  $\mathbb{S}_3$  model is known to be due turning on an abelian flux piercing the  $\mathbb{S}_4$  - matter and Higgs multiplets; and is commonly realised by the splitting spectral method like  $\mathcal{C}_4 = \mathcal{C}_3 \times \mathcal{C}_1$  and  $\mathcal{C}_6 = \tilde{\mathcal{C}}_3 \times \tilde{\mathcal{C}}'_3$ . By using the gauge 2-form field strength  $\mathbf{F}_X$  of the  $U(1)_X$  gauge symmetry considered in [9]; and by thinking of the  $\mathbb{S}_4$  invariance of the  $SO_{10} \times \mathbb{S}_4$  model of table (2.23) in terms of vanishing quantized flux  $\mathcal{F}_\xi^X$  of the 2-form  $\mathbf{F}_X$  over a 2-cycle  $\xi$  in the homology of base of the CY4, namely

$$\begin{aligned} \mathcal{F}_\xi^X|_H &= \int_\xi \mathbf{F}_X = 0 \\ \mathcal{F}_{c_1}^X|_H &= \int_{c_1} \mathbf{F}_X = 0 \quad , \quad H = \mathbb{S}_4; \\ \mathcal{F}_\eta^X|_H &= \int_\eta \mathbf{F}_X = 0 \end{aligned} \quad (3.3)$$

then the breaking of  $\mathbb{S}_4$  monodromy down to discrete subgroups  $H$  may be realised<sup>1</sup> as in table 1 of ref [9] by giving non zero value to  $\mathcal{F}_\xi^X$  as follows

$$\begin{aligned} \mathcal{F}_\xi^X|_H &= N \neq 0 \\ \mathcal{F}_{c_1}^X|_H &= 0 \\ \mathcal{F}_\eta^X|_H &= 0 \end{aligned} \quad , \quad H \subset \mathbb{S}_4 \quad (3.4)$$

where  $N$  is an integer. The above relation is in fact just an equivalent statement of breaking monodromies by using the splitting spectral cover method where covers  $\mathcal{C}_n$  are factorised as product  $\mathcal{C}_{n_1} \times \mathcal{C}_{n_2}$  with  $n_1 + n_2 = n$ . The extra relations  $\mathcal{F}_{c_1}^X|_H = \mathcal{F}_\eta^X|_H = 0$  are the usual conditions to avoid Green- Schwarz mass for the  $U(1)_X$  gauge field potential [2, 3, 9].

In our approach, the effect of the abelian non zero flux  $\mathcal{F}_\xi^X|_H$  is interpreted in terms of

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<sup>1</sup> In table 1 of ref [9], the  $SO_{10} \times \mathbb{S}_4$  model monodromy is broken down to  $\mathbb{Z}_2$ .

piercing the irreducible  $\mathbb{S}_4$ - triplets  $16_i$ ,  $10_i$  and  $10_{[ij]}$  as follows

$$\begin{array}{ccc}
 & & \begin{array}{c|c} \text{curves} & \text{number} \\ \hline \mathbf{16}_a & 3 - N = 2 \\ \mathbf{16}_3 & 3 - 2N = 1 \end{array} \\
 \mathbf{16}_i & \rightarrow & 
 \end{array} \tag{3.5}$$

$$\begin{array}{ccc}
 & & \begin{array}{c|c} \text{curves} & \text{number} \\ \hline \mathbf{10}_a & 2 \\ \mathbf{10}_3 & 1 \end{array} \\
 \mathbf{10}_i & \rightarrow & 
 \end{array} \tag{3.6}$$

$$\begin{array}{ccc}
 & & \begin{array}{c|c} \text{curves} & \text{number} \\ \hline \mathbf{10}_{[a3]} & 2 \\ \mathbf{10}_{[12]} & 1 \end{array} \\
 \mathbf{10}_{[ij]} & \rightarrow & 
 \end{array} \tag{3.7}$$

In geometrical words, flux piercing of the  $16_i$  triplet corresponds to breaking the isotropy of the 3-dimension subspace  $\mathcal{V}_3$  of eq(2.14) as a direct sum  $\mathcal{V}_1 \oplus \mathcal{V}_2$ ; the same breaking happens for the spaces  $\mathcal{V}'_3$  and  $\mathcal{V}''_3$  associated with  $10_i$  and  $10_{[ij]}$ ; and to any non trivial representation of  $\mathbb{S}_4$ . Due to the non zero flux, the  $\mathbb{S}_4$ - triplet  $\{x_1, x_2, x_3\}$  gets splitted into a  $\mathbb{S}_3$ - doublet  $\{|y_1\rangle, |y_2\rangle\}$  and a  $\mathbb{S}_3$ - singlet  $\{|y_3\rangle\}$ ; the new  $y_\mu$  weights are related to the previous weights  $x_\mu$  as follows

$$\begin{aligned}
 y_1 &= \frac{1}{3}(x_1 + x_2 - 2x_3) \\
 y_2 &= \frac{1}{3}(x_1 - 2x_2 + x_3)
 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
 y_3 &= \frac{1}{3}(x_1 + x_2 + x_3) \\
 y_0 &= x_0
 \end{aligned} \tag{3.9}$$

their expressions in terms of the canonical  $t_\mu$ - weights are obtained through the relations  $x_0 = x_0(t_\mu)$ ,  $x_i = x_i(t_\mu)$ . In the  $\{|y_i\rangle\}$  basis, the distribution of the abelian flux among the new directions in the 4-dim permutation module  $\mathcal{V}_4$  is as illustrated on the following  $4 \times 4$  traceless matrix

$$\begin{pmatrix} 4N & & & \\ & -N & & \\ & & -N & \\ & & & -2N \end{pmatrix}_{|y_i\rangle} \tag{3.10}$$

Recall that the permutation group  $\mathbb{S}_3$  has order 6, three conjugacy classes and three irreducible representations;  $6 = 1^2 + 1'^2 + 2^2$ ; its character table is as follows

$\mathfrak{C}_i \backslash \text{irrep } \mathbf{R}_j$	$\chi_I$	$\chi_2$	$\chi_\epsilon$	order
$\mathfrak{C}_1 \equiv e$	1	2	1	1
$\mathfrak{C}_2 \equiv (\alpha\beta)$	1	0	-1	3
$\mathfrak{C}_3 \equiv (\alpha\beta\gamma)$	1	-1	1	2

(3.11)

The group  $\mathbb{S}_3$  has two non commuting generators that can be chosen like  $A = (12)$  and  $B = (123)$  with characters as in above table. Using similar notations as for the case of the group  $\mathbb{S}_4$ ; in particular the property regarding the relation between the characters of reducible  $\mathcal{R}$  and irreducible  $\mathbf{R}_i$  representations of  $\mathbb{S}_3$ , we have

$$\chi_{\mathcal{R}}^{(g)} = n_1 \chi_{\mathbf{R}_1}^{(g)} + n_2 \chi_{\mathbf{R}_2}^{(g)} \quad (3.12)$$

where now  $g = (A, B)$ . Moreover, by using the interpretation of the character  $\chi_{\mathcal{R}}^{(g)}$  in terms of fix points of  $\mathbb{S}_3$ - permutations; we have, on one hand  $\chi_{\mathbf{1} \oplus \mathbf{2}}^{(g)} = (1, 0)$ ; and on the other hand  $\chi_{\mathbf{1}}^{(g)} = (1, 1)$  and  $\chi_{\mathbf{2}}^{(g)} = (0, -1)$ , in agreement with character table (3.11),

$$(1, 0) = (1, 1) + (0, -1) \quad (3.13)$$

Furthermore, by remembering that the  $SO_{10} \times \mathbb{S}_3$ - modeling is done by starting from  $\mathbb{S}_4$ -multiplets and, due to non zero flux, they decompose into irreducible  $\mathbb{S}_3$  representations. For the matter sector of the  $SO_{10} \times \mathbb{S}_3$  model, the four 16-plets transforming in quartet multiplet  $\mathbf{4}$  decomposes in terms of irreducible  $\mathbb{S}_3$  representations as the sum of two singlets  $\mathbf{1}_1$ ,  $\mathbf{1}_2$  and a doublet as follows

$$\mathbf{4} = \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{2} \quad (3.14)$$

However, seen that in  $\mathbb{S}_3$  representation theory, we have two kinds of singlets namely the trivial  $e$  and the sign  $\epsilon$ , we must determine the nature of the singlets  $\mathbf{1}_1$  and  $\mathbf{1}_2$  involved in above equation. While one of these singlets; say the  $\mathbf{1}_1$  should be a trivial singlet as it corresponds just to the weight  $y_0 = x_0$  of eqs(2.15-3.9), we still have to determine the nature of the  $\mathbf{1}_2$ ; but this is also a trivial object since it can be checked directly from our explicit construction as it corresponds precisely to the weight  $y_3$  given by eq(3.9). Nevertheless, this result can be also obtained by using the following consistency relation given by the restriction of the  $\mathbb{S}_4$  relation  $\chi_{\mathbf{4}}^{(G)} = \chi_{\mathbf{1}}^{(G)} + \chi_{\mathbf{3}}^{(G)}$  down to its subgroup  $\mathbb{S}_3$ ; namely

$$\chi_{\mathbf{4}}^{(G)} \Big|_{\mathbb{S}_3} = \chi_{\mathbf{1}}^{(G)} \Big|_{\mathbb{S}_3} + \chi_{\mathbf{3}}^{(G)} \Big|_{\mathbb{S}_3} \quad (3.15)$$

By restricting this relationship to the  $g = (A, B)$  generators of  $\mathbb{S}_3$  and using  $\chi_{\mathbf{3}}^{(g)} = \chi_{\mathbf{1} \oplus \mathbf{2}}^{(g)}$ , the restricted eq(3.15) reads as  $\chi_{\mathbf{1}}^{(g)} + \chi_{\mathbf{1} \oplus \mathbf{2}}^{(g)}$  and leads to

$$(2, 1) = (1, 1) + (0, -1) + (k, l) \quad (3.16)$$

from which we learn that  $(k, l)$  should be equal to  $(1, 1)$ ; and so the  $\mathbf{1}_2$  singlet must be in the trivial representation. A similar reasoning leads to the decompositions of the  $\mathbb{S}_4$ - characters  $\chi_{\mathbf{3}}^{(G)}$  and  $\chi_{\mathbf{3}'}^{(G)}$  down to their  $\mathbb{S}_3$ - counterparts; we find  $\chi_{\mathbf{3}}^{(g)} = \chi_{\mathbf{1} \oplus \mathbf{2}}^{(g)}$  and  $\chi_{\mathbf{3}'}^{(g)} = \chi_{\mathbf{1}' \oplus \mathbf{2}}^{(g)}$  with the value

$$\begin{aligned} \chi_{\mathbf{1} \oplus \mathbf{2}}^{(g)} &= (1, 1) + (0, -1) \\ \chi_{\mathbf{1}' \oplus \mathbf{2}}^{(g)} &= (-1, 1) + (0, -1) \end{aligned} \quad (3.17)$$

Therefore, the  $\mathbb{S}_3$ - matter curves spectrum following from the breaking of (2.23) is given by

matters curves	$\mathbb{S}_3$	homology	$U(1)_X$ flux	character $\chi_{\mathbf{R}}^{(g)}$
$16''_0$	$\mathbf{1}_{+,+}$	$4\xi - c_1$	$4N$	$(1, 1)$
$16''_i$	$\mathbf{2}_{0,-}$	$\eta - 2c_1 - 2\xi$	$-2N$	$(0, -1)$
$16''_3$	$\mathbf{1}_{+,+}$	$-c_1 - 2\xi$	$-2N$	$(1, 1)$
$10''_{[12]}$	$\mathbf{1}_{-,+}$	$2\xi - c_1$	$2N$	$(-1, 1)$
$10''_{[i3]}$	$\mathbf{2}_{0,-}$	$-2c_1 - 2\xi$	$-2N$	$(0, -1)$
$10''_i$	$\mathbf{2}_{0,-}$	$\eta' - 2c_1 - 2\xi$	$-2N$	$(0, -1)$
$10''_3$	$\mathbf{1}_{+,+}$	$2\xi - c_1$	$2N$	$(1, 1)$

(3.18)

where we have indexed the irreducible  $\mathbb{S}_3$ - representations by their characters. In this table,  $\xi$  is a new 2-cycle; and the integer  $N$  standing for the the abelian flux  $\int_{\xi} \mathbf{F}_X$  inducing the breaking down to  $\mathbb{S}_3$ ; see also eq(3.4). Moreover, using properties on characters of tensor products; in particular the typical relations  $X_{p,q} \otimes Y_{k,l} = \sum Z_{n,m}$  requiring  $pk = \sum n$  and  $ql = \sum m$ , we obtain the following  $\mathbb{S}_3$ - algebra

$$\begin{aligned} \mathbf{2}_{0,-} \otimes \mathbf{2}_{0,-} &= \mathbf{1}_{+,+} \oplus \mathbf{1}_{-,+} \oplus \mathbf{2}_{0,-} \\ \mathbf{1}_{p,+} \otimes \mathbf{2}_{0,-} &= \mathbf{2}_{0,-} \\ \mathbf{1}_{p,+} \otimes \mathbf{1}_{+,+} &= \mathbf{1}_{p,+} \\ \mathbf{1}_{+,+} \otimes \mathbf{1}_{-,+} &= \mathbf{1}_{-,+} \end{aligned} \quad (3.19)$$

with  $p = \pm 1$ .

### 3.2 $SO_{10} \times \mathbb{D}_4$ models

We first recall useful aspects on the dihedral group  $\mathbb{D}_4$  and the characters of their representations; then we build three  $SO_{10} \times \mathbb{D}_4$ - models and give their matter and Higgs curves spectrum by using specific properties of the breaking of  $\mathbb{S}_4$  down to  $\mathbb{D}_4$ .

### 3.2.1 Characters in $\mathbb{D}_4$ models

The dihedral  $\mathbb{D}_4$  group is an order 8 subgroup of  $\mathbb{S}_4$  with the usual decomposition property  $8 = 1_1 + 1_2 + 1_3 + 1_4 + 2^2$  showing that, generally speaking,  $\mathbb{D}_4$  has 5 irreducible representations: four of them are 1- dimensional, denoted as  $\mathbf{1}_i$ ; the fifth has 2-dimensions. The finite group  $\mathbb{D}_4$  has two generators  $a$  and  $c$  satisfying the relations  $a^2 = 1$ ,  $c^4 = 1$ , and  $aca = c^3$  with  $c^3 = c^{-1}$  and  $a = a^{-1}$ . It has 5 conjugation classes  $\mathfrak{C}_\alpha$  given by

$$\begin{aligned} \mathfrak{C}_1 &\equiv \{e\} & , & \quad \mathfrak{C}_2 \equiv \{c^2\} & , & \quad \mathfrak{C}_3 \equiv \{c, c^3\} \\ \mathfrak{C}_4 &\equiv \{a, c^2a\} & , & \quad \mathfrak{C}_5 \equiv \{ca, c^3a\} \end{aligned} \quad (3.20)$$

The character table of the irreducible representation of  $\mathbb{D}_4$  is as follows

$\mathfrak{C}_i \backslash \chi_{R_j}$	$\chi_{1_1}$	$\chi_{1_2}$	$\chi_{1_3}$	$\chi_{1_4}$	$\chi_2$	number
$\mathfrak{C}_1$	1	1	1	1	2	1
$\mathfrak{C}_2$	1	1	1	1	-2	1
$\mathfrak{C}_3$	1	1	-1	-1	0	2
$\mathfrak{C}_4$	1	-1	1	-1	0	2
$\mathfrak{C}_5$	1	-1	-1	1	-0	2

(3.21)

To make contact between the three  $(A, B, C) \equiv G$  generators of  $\mathbb{S}_4$  and the two above  $(a, c)$  generators of  $\mathbb{D}_4$ ; notice that the dihedral group has no irreducible 3-cycles; then we have

$$a = A|_{\mathbb{D}_4} \quad c = C|_{\mathbb{D}_4} \quad (3.22)$$

and therefore the following

$\chi_{ij}^{(g)}$	$\chi_{1_1}$	$\chi_{1_2}$	$\chi_{1_3}$	$\chi_{1_4}$	$\chi_2$
$a$	1	-1	1	-1	0
$c$	1	1	-1	-1	0

(3.23)

exhibiting explicitly the difference between the four singlets. For convenience, we denote now on the 5 irreducible representations of the dihedral group by their characters as follows

$$\mathbf{1}_{+,+} \quad , \quad \mathbf{1}_{+,-} \quad , \quad \mathbf{1}_{-,+} \quad , \quad \mathbf{1}_{-,-} \quad , \quad \mathbf{2}_{0,0} \quad (3.24)$$

Notice that from the above character table for the  $(a, c)$  generators of  $\mathbb{D}_4$ , we can build three different kinds of  $\mathbb{D}_4$  algebras; these are:

- $\mathbb{D}_4$ - algebra  $I$

$$\begin{aligned} \mathbf{2}_{0,0} \otimes \mathbf{2}_{0,0} &= \mathbf{1}_{+,+} \oplus \mathbf{1}_{-,-} \oplus \mathbf{1}_{+,-} \oplus \mathbf{1}_{-,+} \\ \mathbf{1}_{p,q} \otimes \mathbf{1}_{p,q} &= \mathbf{1}_{pp',qq'} \\ \mathbf{1}_{p,q} \otimes \mathbf{2}_{0,0} &= \mathbf{2}_{0,0} \end{aligned} \quad (3.25)$$

- $\mathbb{D}_4$ - algebra II

$$\begin{aligned}
\mathbf{2}_{0,0} \otimes \mathbf{2}_{0,0} &= \mathbf{1}_{+,+} \oplus \mathbf{1}_{-,-} \oplus \mathbf{2}_{0,0} \\
\mathbf{1}_{p,q} \otimes \mathbf{1}_{p,q} &= \mathbf{1}_{pp',qq'} \\
\mathbf{1}_{p,q} \otimes \mathbf{2}_{0,0} &= \mathbf{2}_{0,0}
\end{aligned} \tag{3.26}$$

- $\mathbb{D}_4$ - algebra III

$$\begin{aligned}
\mathbf{2}_{0,0} \otimes \mathbf{2}_{0,0} &= \mathbf{1}_{+,-} \oplus \mathbf{1}_{-,+} \oplus \mathbf{2}_{0,0} \\
\mathbf{1}_{p,q} \otimes \mathbf{1}_{p,q} &= \mathbf{1}_{pp',qq'} \\
\mathbf{1}_{p,q} \otimes \mathbf{2}_{0,0} &= \mathbf{2}_{0,0}
\end{aligned} \tag{3.27}$$

To engineer the  $SO_{10} \times \mathbb{D}_4$  models, we proceed as in the case of  $\mathbb{S}_3$ ; we start from the spectrum (2.23) of the  $\mathbb{S}_4$  model and break its monodromy down to  $\mathbb{D}_4$  by turning on an appropriate abelian flux (3.4) that pierces the  $1 + 3$  matter curves  $16_0 \oplus 16_i$  and the  $3 + 3'$  Higgs curves  $10_i \oplus 10_{[ij]}$ . We distinguish three cases depending the way the  $\mathbb{S}_4$ -triplet is splitted; they are described below:

### 3.2.2 Three $SO_{10} \times \mathbb{D}_4$ models

First, we study the two cases where the  $\mathbb{S}_4$ - triplet  $16_i$  and the two  $\mathbb{S}_4$ - triplets  $10_i \oplus 10_{[ij]}$  are completely reduced down in terms of  $\mathbb{D}_4$  singlets. Then, we study the other possible case where these triplets are decomposed as sums of a  $\mathbb{D}_4$ - singlet and a  $\mathbb{D}_4$ - doublet.

#### $\alpha$ ) $SO_{10} \times \mathbb{D}_4$ - models I and II

We will see below that according to the values of the characters of the  $\mathbb{D}_4$ - generators, there are two configurations for the splitting  $3 = 1 + 1 + 1$ ; they depend on presence or absence of trivial  $1_{++}$  representation as given below:

- a splitting with no  $\mathbb{D}_4$ - trivial singlet

$$\mathbf{3} = \mathbf{1}_{+,-} \oplus \mathbf{1}_{+,-} \oplus \mathbf{1}_{-,+} \tag{3.28}$$

- a splitting with a  $\mathbb{D}_4$ - trivial singlet

$$\mathbf{3} = \mathbf{1}_{+,+} \oplus \mathbf{1}_{-,-} \oplus \mathbf{1}_{+,-} \tag{3.29}$$

To establish this claim, we start from the character of the generators of  $\mathbb{S}_4$ ; and consider the computation of the following restriction down to  $\mathbb{D}_4$  subgroup

$$\chi_4^{(g)} \Big|_{\mathbb{D}_4} = \chi_1^{(g)} \Big|_{\mathbb{D}_4} + \chi_3^{(g)} \Big|_{\mathbb{D}_4} \tag{3.30}$$

Thinking of the triplet  $\mathbf{3}$  as the direct sum of three singlets  $\mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3$ , we then have the following character relationship

$$\chi_{\mathbf{4}}^{(g)} \Big|_{\mathbb{D}_4} = \chi_{\mathbf{1}_1}^{(g)} \Big|_{\mathbb{D}_4} + \chi_{\mathbf{1}_2}^{(g)} \Big|_{\mathbb{D}_4} + \chi_{\mathbf{1}_3}^{(g)} \Big|_{\mathbb{D}_4} + \chi_{\mathbf{1}_4}^{(g)} \Big|_{\mathbb{D}_4} \quad (3.31)$$

By combining eqs(2.22) and (3.22), we learn that the left hand side of equation is nothing but  $\chi_{\mathbf{4}}^{(g)} \Big|_{\mathbb{D}_4} = (2, 0)$ ; while, by using the characters of the singlets of table (3.21), the right hand side decomposes like

$$(2, 0) = (1, 1) + (k_1, l_1) + (k_2, l_2) + (k_3, l_3) \quad (3.32)$$

with  $k_i = \pm 1$  and  $l_i = \pm 1$ . Explicitly

$$\begin{aligned} k_1 + k_2 + k_3 &= +1 \\ l_1 + l_2 + l_3 &= -1 \end{aligned} \quad (3.33)$$

which can be solved in two manners: (i) either as

$$(2, 0) = (1, 1) + (1, -1) + (1, -1) + (-1, 1) \quad (3.34)$$

involving three kinds of irreducibles 1-dim representations of  $\mathbb{D}_4$ ; the trivial representation with character  $(1, 1)$  and two others with characters  $(-1, 1)$  appearing once and the  $(1, -1)$  appearing twice; or (ii) like

$$(2, 0) = (1, 1) + (1, -1) + (1, 1) + (-1, -1) \quad (3.35)$$

where the trivial representation appears twice. Accordingly, we have the following curves spectrums:

•  $SO_{10} \times \mathbb{D}_{4^-}$  model I

It is given by the decomposition (3.34); its matter spectrum reads as

matters curves	$\mathbb{D}_4$	homology	$U(1)_x$ flux	character $\chi_{\mathbf{R}}^{(g)}$	(3.36)
$\mathbf{16}''_0$	$\mathbf{1}_{+,+}$	$4\xi - c_1$	$4N$	$(1, 1)$	
$\mathbf{16}''_1$	$\mathbf{1}_{+,-}$	$\eta - c_1 - \xi$	$-N$	$(1, -1)$	
$\mathbf{16}''_2$	$\mathbf{1}_{+,-}$	$-c_1 - \xi$	$-N$	$(1, -1)$	
$\mathbf{16}''_3$	$\mathbf{1}_{-,+}$	$-c_1 - 2\xi$	$-2N$	$(-1, 1)$	
$\mathbf{10}''_{[12]}$	$\mathbf{1}_{-,+}$	$2\xi - c_1$	$2N$	$(-1, 1)$	
$\mathbf{10}''_{[13]}$	$\mathbf{1}_{+,-}$	$-c_1 - \xi$	$-N$	$(1, -1)$	
$\mathbf{10}''_{[23]}$	$\mathbf{1}_{+,-}$	$-c_1 - \xi$	$-N$	$(1, -1)$	
$\mathbf{10}''_1$	$\mathbf{1}_{+,-}$	$\eta' - 2c_1 - \xi$	$-N$	$(1, -1)$	
$\mathbf{10}''_2$	$\mathbf{1}_{-,+}$	$-c_1 - \xi$	$-N$	$(-1, 1)$	
$\mathbf{10}''_3$	$\mathbf{1}_{-,+}$	$2\xi - c_1$	$2N$	$(-1, 1)$	

and

flavons	character $\chi_{\mathbf{R}}^{(g)}$
$\mathbf{1}_{-,+}$	$(-1, 1)$
$\mathbf{1}_{+,-}$	$(1, -1)$
$\mathbf{1}_{+,-}$	$(1, -1)$
$\mathbf{1}_{+,-}$	$(1, -1)$
$\mathbf{1}_{-,+}$	$(-1, 1)$
$\mathbf{1}_{-,+}$	$(-1, 1)$

(3.37)

together with their adjoints.

•  $SO_{10} \times \mathbb{D}_4$ - model II

It is given by the decomposition (3.35) with matter curve spectrum like

matters curves	$\mathbb{D}_4$	homology	$U(1)_X$ flux	character $\chi_{\mathbf{R}}^{(g)}$
$\mathbf{16}_0''$	$\mathbf{1}_{+,+}$	$4\xi - c_1$	$4N$	$(1, 1)$
$\mathbf{16}_1''$	$\mathbf{1}_{+,-}$	$\eta - c_1 - \xi$	$-N$	$(1, -1)$
$\mathbf{16}_2''$	$\mathbf{1}_{-,-}$	$-c_1 - \xi$	$-N$	$(-1, -1)$
$\mathbf{16}_3''$	$\mathbf{1}_{+,+}$	$-c_1 - 2\xi$	$-2N$	$(1, 1)$
$\mathbf{10}_{[12]}''$	$\mathbf{1}_{+,-}$	$2\xi - c_1$	$2N$	$(1, -1)$
$\mathbf{10}_{[13]}''$	$\mathbf{1}_{+,+}$	$-c_1 - \xi$	$-N$	$(1, 1)$
$\mathbf{10}_{[23]}''$	$\mathbf{1}_{-,-}$	$-c_1 - \xi$	$-N$	$(-1, -1)$
$\mathbf{10}_1''$	$\mathbf{1}_{-,+}$	$\eta' - 2c_1 - \xi$	$-N$	$(-1, 1)$
$\mathbf{10}_2''$	$\mathbf{1}_{-,-}$	$-c_1 - \xi$	$-N$	$(-1, -1)$
$\mathbf{10}_3''$	$\mathbf{1}_{+,+}$	$2\xi - c_1$	$2N$	$(1, 1)$

(3.38)

and

flavons	character $\chi_{\mathbf{R}}^{(g)}$
$\mathbf{1}_{+,-}$	$(1, -1)$
$\mathbf{1}_{+,+}$	$(1, 1)$
$\mathbf{1}_{-,-}$	$(-1, -1)$
$\mathbf{1}_{-,+}$	$(-1, 1)$
$\mathbf{1}_{-,-}$	$(-1, -1)$
$\mathbf{1}_{+,+}$	$(1, 1)$

(3.39)

$\beta$ )  $SO_{10} \times \mathbb{D}_4$ - model III

This model corresponds to the splitting  $\mathbf{3} = \mathbf{1} \oplus \mathbf{2}$ ; the restricted character relation (3.30)



decomposes like

$$\chi_{\mathbf{4}}^{(g)} \Big|_{\mathbb{D}_4} = \chi_{\mathbf{1}}^{(g)} + \chi_{\mathbf{1}'}^{(g)} + \chi_{\mathbf{2}}^{(g)} \quad (3.40)$$

leading to

$$(2, 0) = (1, 1) + (k_1, l_1) + (k_2, l_2) \quad (3.41)$$

and then  $(k_1, l_1) + (k_2, l_2) = (1, -1)$ . However seen that the character  $\chi_{\mathbf{2}}^{(g)} = (0, 0)$ ; it follows that

$$\mathbf{3} = \mathbf{2}_{0,0} \oplus \mathbf{1}_{+,-} \quad (3.42)$$

Therefore the matter spectrum of the  $SO_{10} \times \mathbb{D}_4$ - model III is given by

matters curves	$\mathbb{D}_4$	homology	$U(1)_X$ flux	character $\chi_{\mathbf{R}}^{(g)}$
$\mathbf{16}_0''$	$\mathbf{1}_{+,+}$	$4\xi - c_1$	$4N$	$(1, 1)$
$\mathbf{16}_i''$	$\mathbf{2}_{0,0}$	$\eta - 2c_1 - 2\xi$	$-2N$	$(0, 0)$
$\mathbf{16}_3''$	$\mathbf{1}_{+,-}$	$-c_1 - 2\xi$	$-2N$	$(1, -1)$
$\mathbf{10}_{[12]}''$	$\mathbf{1}_{-,+}$	$2\xi - c_1$	$2N$	$(-1, 1)$
$\mathbf{10}_{[23]}''$	$\mathbf{2}_{0,0}$	$-2c_1 - 2\xi$	$-2N$	$(0, 0)$
$\mathbf{10}_i''$	$\mathbf{2}_{0,0}$	$\eta' - 2c_1 - 2\xi$	$-2N$	$(0, 0)$
$\mathbf{10}_3''$	$\mathbf{1}_{+,-}$	$2\xi - c_1$	$2N$	$(1, -1)$

(3.43)

and

flavons	character $\chi_{\mathbf{R}}^{(g)}$
$\mathbf{1}_{-,+}$	$(-1, 1)$
$\mathbf{2}_{0,0}$	$(0, 0)$
$\mathbf{2}_{0,0}$	$(0, 0)$
$\mathbf{1}_{+,-}$	$(1, -1)$

(3.44)

Notice that in tables (3.36-3.43), the homology classes of the new curves are obtained as usual by requiring the sum of their homology classes to be equal to the class of their mother matter curve in  $\mathbb{S}_4$  model. The extra 2-cycle class  $\xi$  and the corresponding integral flux  $N = \int_{\xi} \mathbf{F}_X$  are associated with the breaking of  $\mathbb{S}_4$  down to  $\mathbb{D}_4$ ; the 2-form  $\mathbf{F}_X$  is as in eq(3.4).

## 4 Conclusion and discussions

In this work, we have used characters of discrete group representations to approach the engineering of  $SO_{10} \times \Gamma$  models with discrete monodromy  $\Gamma$  contained in  $\mathbb{S}_4$ ; this method generalises straightforwardly to  $SU_5 \times \Gamma$  and its breaking down to MSSM. In

this construction, curves of the GUT- models are described by the characters  $\chi_{\mathbf{R}_i}$  of the irreducible representations  $\mathbf{R}_i$  of the discrete group  $\Gamma$ .

After having introduced the idea of the character based method; we have revisited the study of the  $\mathbb{S}_4$ - and  $\mathbb{S}_3$ - models from the view of monodromy irreducible representations and their characters; see table (2.23) for the case  $\Gamma = \mathbb{S}_4$ , the table (3.18) for  $\mathbb{S}_3$ - model. Then, we have extended the construction to the dihedral group where we have found that there are three  $SO_{10} \times \mathbb{D}_4$ - models with curves spectrum as in tables (3.36), (3.38) and (3.43).

The approach introduced and developed in this study has two remarkable particularities: (i) first it allows to build GUT- models with subgroups inside the  $\mathbb{S}_4$  permutation symmetry like  $\mathbb{D}_4$  and  $\mathbb{A}_4$  without resorting to the use of Galois theory. The latter is known to lead to non linear constraints on the holomorphic sections of the spectral covers; and requires more involved tools for their solutions. (ii) second, it is based on a natural quantity of discrete symmetry groups; namely the characters of their representations; known as basic objects to deal with finite order symmetry groups.

#### *Building superpotentials using characters*

The character based method developed in this study can be also used to build superpotentials. For the case of  $SO_{10} \times \mathbb{D}_4$ -models; superpotentials  $W$  invariant under discrete symmetry  $\mathbb{D}_4$  are obtained by requiring their character as  $\chi_{\mathbf{R}}^{(g)}(W) = (1, 1)$ . By focussing on the models (3.36-3.38); and denoting the matter curves with  $(p, q)$  character as  $\mathbf{16}_{p,q}$  and the Higgs curve with character  $(\alpha, \beta)$  like  $\mathbf{10}_{\alpha,\beta}$ ; and restricting to typical tri-coupling superpotential, we have the following candidate

$$W = \sum \lambda_{pq}^{p'q'} \mathbf{16}_{p,q} \otimes \mathbf{16}_{p',q'} \otimes \mathbf{10}_{1/pp', 1/qq'}$$

More general expressions can be written down by implementing flavons as well. However, to write down phenomenologically acceptable Yukawa couplings that agree with low energy effective field constraints such as reproducing a MSSM like spectrum and suppressing rapid proton decay, one must have a diagonal tree-level Yukawa coupling for the heaviest top-quark family; one also need to introduce extra discrete symmetries such as R-parity or the  $\mathbb{Z}_2$  geometric symmetry of [19, 9, 12] to rule out undesired couplings. Diagonal tree level 3-couplings for the top-quark family have been studied in [9] for the case of  $\mathbb{S}_3$  and its subgroups; they extend directly to our analysis; it reads in character language as

$$W_{tree} = \lambda_{top} \mathbf{16}_{+-} \otimes \mathbf{16}_{+-} \otimes \mathbf{10}_{++}$$

singling out the spectrum of the  $\mathbb{D}_4$ - model (3.38). For the spectrum (3.43), a superpotential with a diagonal Yukawa coupling for a top-quark in a  $\mathbb{D}_4$ -singlet requires flavons;

if taking the top-quark matter curve in  $\mathbf{16}_{+-}$  as above; a superpotential  $W_*$  candidate would have the form

$$W_* \sim \mathbf{16}_{+-} \otimes \mathbf{16}_{+-} \otimes \mathbf{10}_{-+} \otimes \mathbf{1}_{-+}$$

a similar conclusion is valid if taking the top-quark in  $\mathbf{16}_{++}$ .

#### *Breaking symmetries in $SO_{10} \times \mathbb{D}_4$ - models*

The descent from  $SO_{10} \times \mathbb{D}_4$ - theory to  $SU_5 \times \Gamma$ -models with discrete symmetries  $\Gamma \subset \mathbb{D}_4$  follows by using  $U(1)_X$  flux to pierce the matter  $\mathbf{16}$  and Higgs  $\mathbf{10}$  curves as done in [9]. For matter sector for example, we start from  $M_{16}$  multiplets  $\{\mathbf{16}_i\}_{M_{16}}$  and use  $N$  flux units of  $U(1)_X$  to pierce the curve package; and decompose it like  $\{\mathbf{10}\}_{M_{16}} \oplus \{\mathbf{5}\}_{M_{16}-N} \oplus \{\mathbf{1}\}_{M_{16}+N}$ ; similar decompositions are valid for the Higgs sector; explicit MSSM- like models using discrete group character approach will be given in [17]. One can also break the  $\mathbb{D}_4$  monodromy group down to  $\mathbb{Z}_4$  subgroup by using flux and following the same character based method described in this paper.

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